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Kuba, Markus; Sulzbach, Henning

DOI:
10.1017/jpr.2016.89

Citation for published version (Harvard):
Kuba, M & Sulzbach, H 2017, 'On martingale tail sums in affine two-color urn models with multiple drawings', *Journal of Applied Probability*, vol. 54, no. 1, pp. 96-117. https://doi.org/10.1017/jpr.2016.89

ON MARTINGALE TAIL SUMS IN AFFINE TWO-COLOR URN MODELS WITH MULTIPLE DRAWINGS

MARKUS KUBA AND HENNING SULZBACH [1]

ABSTRACT. In two recent works, Kuba and Mahmoud (arXiv:1503.090691 and arXiv:1509.09053) introduced the family of two-color affine balanced Pólya urn schemes with multiple drawings. We show that, in large-index urns (urn index between $1/2$ and 1) and triangular urns, the martingale tail sum for the number of balls of a given color admits both a Gaussian central limit theorem as well as a law of the iterated logarithm. The laws of the iterated logarithm are new even in the standard model when only one ball is drawn from the urn in each step (except for the classical Pólya urn model). Finally, we prove that the martingale limits exhibit densities (bounded under suitable assumptions) and exponentially decaying tails. Applications are given in the context of node degrees in random linear recursive trees and random circuits.

1. INTRODUCTION

Pólya urn schemes are useful mathematical toy models for growth processes with a wide range of applications in several areas including the analysis of random trees, graphs and algorithms, population genetics and the spread of epidemics. For a discussion of these and further applications, we refer to the monographs of Johnson and Kotz [26] and Mahmoud [34]. Instances of urn models with multiple drawings were first discussed by Mahmoud and Tsukiji [45] in the context of random circuits. The model was then further developed in several recent contributions by Johnson, Kotz and Mahmoud [27], Chen and Wei [7], Renlund [41], Mahmoud [35], Moler, Plo and Urmeneta [37], Chen and Kuba [8], Kuba, Mahmoud and Panholzer [31] and Kuba and Mahmoud [32, 33].

Let us describe the details of the evolution of the two-color urn process. For convenience, we use the colors white and black. In each step, we take a sample of $m \geq 1$ balls from the urn. Here we distinguish two scenarios: in model $\mathcal{M}$, balls are drawn without replacement; whereas, in model $\mathcal{R}$, the sample is obtained with replacement. The pick is then put back in the urn together with a certain number of additional white and black balls determined as follows: given that the sample contained k white and $m-k$ black balls, we add a_{m-k} white and b_{m-k} black balls. Here, $a_k, b_k, 0 \leq k \leq m$ are integers, where negative values are allowed and correspond to removing balls from the urn. By $\mathbf{M}$, we denote the ball replacement matrix of this process,

$$\mathbf{M} = \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ \vdots & \vdots \\ a_{m-1} & b_{m-1} \\ a_m & b_m \end{pmatrix}. \quad (1)$$

As usual, W_n and B_n describe the number of white and black balls in the urn after n draws. Further, we let $T_n = W_n + B_n$ be the total number of balls at time n. An urn scheme is called *balanced* if, at each step, the total number of added balls $\sigma \geq 1$ is constant. In other words, $a_k + b_k = \sigma$, $0 \leq k \leq m$. In balanced urns, we have, almost surely, $T_n = T_0 + \sigma n$. We call the scheme *tenable*, if, almost surely, the process of drawing balls and updating the urn configuration can be continued forever. Throughout the work, we only consider balanced and tenable urn models. We also assume that both W_0 and B_0 are deterministic.

Date: November 28, 2016.

2000 *Mathematics Subject Classification.* 60F15, 60C05, 60F05, 60G42.

Key words and phrases. Urn model, martingale central limit theorem, law of the iterated logarithm, large-index urns, triangular urns.

[1]This work was supported by a Feodor Lynen Research Fellowship from the Alexander von Humboldt Foundation.

1.1. **Affine models.** For $m = 1$, the classical analysis of the composition of Pólya urns is based on the observation that, for suitable real-valued sequences $\alpha_n, \beta_n, n \geq 1$, the conditional expectation of W_n exhibits the following affine structure

$$\mathbb{E}\big[W_n \,|\, \mathcal{F}_{n-1}\big] = \alpha_n W_{n-1} + \beta_n, \qquad n \geq 1. \tag{2}$$

Here, $\mathcal{F}_n$ denotes the σ-algebra generated by the first n draws from the urn. In urn schemes with multiple drawings, the conditional expectation of W_n generally involves higher powers of W_{n-1} which complicates the situation drastically. We believe that the analysis of the general model requires new techniques and do not approach this problem here. Following [32, 33], in order to maintain the familiar structure (2) from the case $m = 1$, we call an urn scheme with multiple drawings *affine* if (2) is satisfied for some deterministic sequences $\alpha_n, \beta_n, n \geq 1$. By [32, Proposition 1], balanced urn models are affine if and only if the entries in the first column of the replacement matrix $\mathbf{M}$ satisfy the recurrence:

$$a_k = (m-k)(a_{m-1} - a_m) + a_m, \qquad \text{for } \ 0 \leq k \leq m.$$

It follows that, in affine models, the matrix $\mathbf{M}$ can be expressed only in terms of the parameters a_{m-1}, a_m, and the balance factor σ. Similarly, α_n and β_n in (2) can be given in terms of a_{m-1}, a_m and σ (see [32, Proposition 1]) which we do not repeat here since α_n and β_n are of no relevance in our work.

In the case $m = 1$, it is well-known that the eigenvalues of the replacement matrix play an important role in the classification of different urn schemes. This transfers directly to the case of multiple drawings upon defining $\Lambda_1 \geq \Lambda_2$ to be the two eigenvalues of the submatrix $\begin{pmatrix} a_{m-1} & b_{m-1} \\ a_m & b_m \end{pmatrix}$, compare Theorem 1 below. We define the *urn index* by

$$\Lambda := \frac{m\Lambda_2}{\Lambda_1} = \frac{m}{\sigma}(a_{m-1} - a_m) \in (-\infty, 1]. \tag{3}$$

As for $m = 1$, urn schemes can be divided in the following three fundamentally different cases:

i) Urn schemes with $a_m \geq 1, b_0 \geq 1$ and *small* index $\Lambda \leq 1/2$, the case $\Lambda = \frac{1}{2}$ being critical,
ii) Urn schemes with $a_m \geq 1, b_0 \geq 1$ and *large* index $\frac{1}{2} < \Lambda < 1$,
iii) *Triangular* urn models with $a_m = 0$ or $b_0 = 0$ and arbitrary $0 < \Lambda \leq 1$. The special case $\Lambda = 1$ corresponds to the so-called *generalized* Pólya urn model introduced in [7, 8]. Here, $a_m = b_0 = 0$.

In the context of triangular urn models, the relation $B_n = T_n - W_n$ allows us to restrict ourselves to urns with $a_m = 0$ and $b_0 \geq 0$. For small-index urns, we always exclude the case $\Lambda = 0$ from the results since it implies a deterministic evolution of the urn composition. Similarly, in triangular urns, we always assume $W_0 \geq 1$, and, additionally, $B_0 \geq 1$ if $\Lambda = 1$.

1.2. **Known results.** The main results of the two works [32, 33] can be summarized in the following theorem which is well-known and classical for $m = 1$. Here, and throughout the work, we abbreviate

$$\zeta = \frac{a_m}{1 - \Lambda}, \qquad Q = \frac{\Gamma(\frac{T_0}{\sigma} + \Lambda)}{\Gamma(\frac{T_0}{\sigma})}. \tag{4}$$

Note that, tenability of the scheme depends on the matrix $\mathbf{M}$, the initial configuration (W_0, B_0) and the model under consideration. For a discussion, see Lemma 1 in Section 3.1.

Theorem 1. *Let W_n be the number of white balls at time n in an affine balanced and tenable two-color urn model (model $\mathcal{M}$ or model $\mathcal{R}$) with replacement matrix $\mathbf{M}$ given in (1) and fixed initial configuration (W_0, B_0).*

i) For small-index urns with $0 \neq \Lambda \leq 1/2$[1], we have, in distribution,

$$\frac{W_n - \zeta n}{\sqrt{n\ell_n}} \to \mathcal{N}(0, \gamma_1^2), \qquad \ell_n = \begin{cases} 1 & \text{if } \Lambda < 1/2, \\ \log n & \text{if } \Lambda = 1/2. \end{cases}$$

[1] In [32, Theorem 3], the authors impose the additional condition $T_0 + m(a_{m-1} - a_m) > 0$. It is not hard to see that the result holds without making this assumption. In fact, the relevant argument is stated on page 5 in [32].

Here, $\mathcal{N}(0, \gamma_1^2)$ *denotes a zero-mean normal random variable with variance* $\gamma_1^2 > 0$ *where*

$$\gamma_1 = \begin{cases} \frac{\Lambda}{1-\Lambda}\sqrt{\frac{a_m b_0}{m(1-2\Lambda)}} & \text{if } \Lambda < 1/2, \\ \sqrt{\frac{a_m b_0}{m}} & \text{if } \Lambda = 1/2. \end{cases} \tag{5}$$

ii) For large-index urns with $1/2 < \Lambda < 1$*, we have, almost surely,*

$$\frac{Q \cdot (W_n - \zeta n)}{n^\Lambda} \to \mathcal{W}_\infty, \quad \mathbb{E}[\mathcal{W}_\infty] = 0.$$

iii) For triangular urns with $0 < \Lambda \le 1$ *and* $a_m = 0$*, we have, almost surely,*

$$\frac{Q \cdot W_n}{n^\Lambda} \to \mathfrak{W}_\infty, \quad \mathbb{E}[\mathfrak{W}_\infty] = W_0.$$

The convergence for large-index and triangular urns holds with respect to all moments and the random variables $\mathcal{W}_\infty, \mathfrak{W}_\infty$ *are not almost surely constant.*

We shortly discuss the theorem in the classical case $m = 1$. The central limit theorems for small-index and large-index balanced urns go back to Athreya and Karlin [1] under the assumption that $a_0, b_1 \ge -1$, the important case of Friedman's urn had earlier been solved earlier by Freedman [16]. Bagchi and Pal [2] proved the Gaussian central limit theorem in small-index urns in the general case using the method of moments. By similar techniques as adopted in this work, Gouet [17] showed functional central limit theorems covering the statement for small- and large-index urns as well as for triangular urns. Variants of Theorem 1 *i), ii)* have been obtained based on substantially different techniques: Flajolet, Gabarró and Pekari [14] and Flajolet, Dumas and Puyhaubert [13] used singularity analysis, Pouyanne [40] applied purely algebraic methods in the context of large-index urns, and, very recently, Neininger and Knape [30] worked out an approach based on the contraction method. Janson's comprehensive work [23] based on a strengthening of the ideas in [1] also treats certain non-balanced urn models and contains an elaborate summary of works in the context of Theorem 1 *i), ii)*. Properties of the law of the martingale limit $\mathcal{W}_\infty$ such as characteristic functions, densities, moments and characterizing stochastic fixed-point equations were studied by Chauvin, Pouyanne and Sahnoun [5], Neininger and Knape [30], as well as by Chauvin, Pouyanne and Mailler [6]. Similarly, in triangular urn schemes, the law of $\mathfrak{W}_\infty$ was studied by Janson [24, 25] and in [14]. Note that, [24] contains a full characterization of the limiting distributions in triangular urns covering all cases of zero-balanced and unbalanced schemes. For more references and results on large deviations and convergence rates, we refer to the discussions in the literature cited.

For general $m \ge 1$, explicit expressions for the (positive integer) moments of the non-normal limits in large-index and triangular urns, that is $\mathfrak{W}_\infty$ and $\mathcal{W}_\infty$ in Theorem 1, have been obtained in [33]. It is important to note that the structure of higher moments for multiple drawings $m > 1$ is significantly more involved compared to the case $m = 1$ where simplifications occur (see also the discussion in [8]). Note that the main results of this work show that these limiting distributions have exponentially small tails.

1.3. **Aim of the paper.** For a martingale (Y_n) converging almost surely to a random variable Y_∞, the sequence $(Y_n - Y_\infty)$ is called *martingale tail sum*. A classical result for urn models is the central limit theorem and the law of the iterated logarithm by Heyde [22] for the martingale tail sum in the original Pólya urn model with sample size $m = 1$. Again, for $m = 1$, central limit theorems for the tail sums in balanced small- and large-index urns as well as in triangular urns are contained in the functional limit theorems in [17]. A corresponding law of the iterated logarithm for small-index urns was given by Bai, Hu and Zhang [3]. A classical result for urn models is the central limit theorem and the law of the iterated logarithm by Heyde [22] for the martingale tail sum in the original Pólya urn model with sample size $m = 1$. Again, for $m = 1$, central limit theorems for the tail sums in balanced small- and large-index urns as well as in triangular urns are contained in the functional limit theorems in [17]. A corresponding law of the iterated logarithm for small-index urns was given by Bai, Hu and Zhang [3]. Recently, several articles analyzing related random discrete structures have been devoted to martingale tails sums: Móri [38] obtained a central limit theorem for maximum degree of the plane-oriented recursive trees. Neininger [29] proved a central limit theorem for the martingale tail sum of Régnier's martingale for the path length in random binary search trees. Fuchs [15] reproved this result using the method of moments; a refinement of this result is given by Grübel and Kabluchko [19]. Sulzbach [43] generalized the result for binary search trees to a family of increasing trees containing amongst others binary search trees, recursive trees and plane-oriented recursive trees, also obtaining a law of the iterated logarithm.

In this work we derive central limit theorems and laws of the iterated logarithm for the martingale tail sums arising in large-index and triangular affine urn models for general $m \geq 1$. Note that, even for $m = 1$, the laws of the iterated logarithm are new except for the case $\Lambda = 1$. We also extend the results in [7, 8] to show that the martingale limits in all models admit densities and exponentially small tails. Throughout the work, we exclusively use tools from discrete-time martingale theory as summarized in Sections 3.4 and 3.5. For $m = 1$, the urn model allows for a continuous-time multi-type branching process embedding [1, 23] as well as for recursive distributional decompositions [30, 6] leading to much stronger results about the limiting random variables. Since these techniques seem not to be directly applicable for $m > 1$, we leave it as an open problem to derive more precise information about the limit laws such as infinite divisibility, smoothness of densities, unbounded support or characterizations based on stochastic-fixed point equations.

1.4. **Notation.** We denote by $x^{\underline{k}}$ the kth falling factorial, $x(x-1)\dots(x-k+1)$, $k \geq 0$, with $x^{\underline{0}} = 1$. For real-valued sequences a_n, b_n, we write $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$ (almost surely, if the sequences are random). Further, we use the big-$\mathcal{O}$ Landau notation for sequences as $n \to \infty$. By $\mathcal{N}$ we denote a standard normal random variable. We call a random variable X *Subgaussian* if there exist $c, C > 0$, such that,

$$\mathbb{P}(|X| \geq t) \leq Ce^{-ct^2}, \quad t > 0.$$

By $\mathbf{bin}(n, p)$ we denote the binomial distribution with parameters $n \in \mathbb{N}$ (number of trials) and $p \in [0, 1]$ (success probability). Similarly, we use $\mathbf{hyp}(N, K, m)$ to denote the hypergeometric distribution counting the number of white balls in a sample of size $m \in \mathbb{N}$ balls taken from urn containing $N \geq m$ balls, $m \leq K \leq N$ among them being white.

2. Results

Our main results concern the asymptotic behavior of W_n in large-index urns and triangular urns. Our results cover both model $\mathcal{M}$ and model $\mathcal{R}$. We refer to Lemma 2 for an explicit formula for $\mathbb{E}[W_n]$. Let

$$g_n = \frac{\binom{n-1+\frac{T_0}{\sigma}}{n}}{\binom{n-1+\frac{T_0}{\sigma}+\Lambda}{n}} = Qn^{-\Lambda}\Big(1 + O\Big(\frac{1}{n}\Big)\Big), \quad n \geq 1, \tag{6}$$

with Q given in (4) and Λ given in (3).

Theorem 2 (Large-index urns)**.** *Let W_n be the number of white balls at time n in a large-index urn with $1/2 < \Lambda < 1$. Then, $\mathcal{W}_n := g_n(W_n - \mathbb{E}[W_n])$ is an almost surely convergent martingale. Its limit $\mathcal{W}_\infty$ is Subgaussian and admits a bounded density on $(-\infty, \infty)$. In distribution and with convergence of all moments,*

$$\alpha n^{\Lambda-1/2}(\mathcal{W}_n - \mathcal{W}_\infty) \to \mathcal{N}, \quad \alpha = \frac{(1-\Lambda)}{Q\Lambda}\sqrt{\frac{m(2\Lambda-1)}{a_m b_0}}. \tag{7}$$

Almost surely,

$$\limsup_{n\to\infty} \frac{\alpha n^{\Lambda-1/2}(\mathcal{W}_n - \mathcal{W}_\infty)}{\sqrt{2\log\log n}} = 1, \quad \liminf_{n\to\infty} \frac{\alpha n^{\Lambda-1/2}(\mathcal{W}_n - \mathcal{W}_\infty)}{\sqrt{2\log\log n}} = -1.$$

Theorem 3 (Triangular urns)**.** *Let W_n be the number of white balls at time n in a triangular urn with $\Lambda \leq 1$ and $a_m = 0$. Then, $\mathfrak{W}_n := g_n W_n$ is an almost surely convergent martingale. Its limit $\mathfrak{W}_\infty$ admits a density on $(0, \infty)$ which is bounded if $W_0 \geq a_{m-1}, \Lambda < 1$ or $W_0, B_0 \geq a_{m-1}, \Lambda = 1$. For $\Lambda > 1/2$, it is Subgaussian, where $\mathcal{W}_\infty \leq T_0$ for $\Lambda = 1$. For $\Lambda \leq 1/2$, it has a finite momentum-generating function in some non-empty open interval containing zero. In distribution (and with convergence of all moments in the second display for $\Lambda > 1/2$) ,*

$$\beta\eta n^{\Lambda/2}(\mathfrak{W}_n - \mathfrak{W}_\infty) \to \mathcal{N}, \quad \beta n^{\Lambda/2}(\mathfrak{W}_n - \mathfrak{W}_\infty) \to (\eta')^{-1}\mathcal{N}, \tag{8}$$

where $\eta', \mathcal{N}$ are independent and η' is distributed like η given in (9). Almost surely,

$$\limsup_{n\to\infty} \frac{\beta\eta n^{\Lambda/2}(\mathfrak{W}_n - \mathfrak{W}_\infty)}{\sqrt{2\log\log n}} = 1, \quad \liminf_{n\to\infty} \frac{\beta\eta n^{\Lambda/2}(\mathfrak{W}_n - \mathfrak{W}_\infty)}{\sqrt{2\log\log n}} = -1.$$

Here,

$$\beta = \begin{cases} \sqrt{\frac{W_0}{a_{m-1}}} & \text{if } \Lambda < 1, \\ \sqrt{m\mathbb{E}[\mathfrak{W}_\infty(T_0 - \mathfrak{W}_\infty)]} & \text{if } \Lambda = 1. \end{cases} \quad \eta = \begin{cases} \mathfrak{W}_\infty^{-1/2} & \text{if } \Lambda < 1, \\ (\mathfrak{W}_\infty(T_0 - \mathfrak{W}_\infty))^{-1/2} & \text{if } \Lambda = 1. \end{cases} \tag{9}$$

Let us give a detailed discussion of the results. For $m > 1$, almost sure convergence of the martingales were established in [32], compare also Theorem 1, where special cases had been considered earlier [7, 8, 45]. Asymptotic statements about the martingale tail sums are novel for $m > 1$ and so are the properties of the limiting distribution with the exception of the generalized Pólya urn studied in [7, 8]. For $m = 1$, the laws of iterated logarithm for large-index and triangular urns with $\Lambda < 1$ are new; for the remaining results, compare the discussion in the introduction. For $m = 1$, the moments of $\mathfrak{W}_\infty$ exhibit simple explicit expressions [24, Theorem 1.7]. From these results, it is easy to see that Theorem 3 is optimal in the sense that, for $m = 1$ and $\Lambda < 1/2$, $\mathfrak{W}_\infty$ is not Subgaussian[2]. In the critical case $\Lambda = 1/2$, we believe that $\mathfrak{W}_\infty$ is Subgaussian for any $m > 1$ but our methods are not sufficiently strong to deduce this. Similarly, by the results in [25, Section 9], for $m = 1$, the density of $\mathfrak{W}_\infty$ is unbounded if $\Lambda < 1, W_0 < c$ or $\Lambda = 1, \min(W_0, B_0) < c$. Finally, note that the first convergence in (8) is mixing in the sense of Rényi and Révész [42]. This property allows to deduce the second convergence.

For the sake of completeness, we briefly discuss balanced affine small-index urns with $0 \neq \Lambda \leq 1/2$. Recall the martingale central limit theorem for W_n in Theorem 1 *i)* which is Theorem 3 in [32]. Note that, by Corollary 1 in [20], this convergence is with respect to all moments. Further, analogously to the results presented in this work and relying on the same martingale methods, that is Theorem 1, Corollary 1 and Corollary 2 in [22], the following laws of the iterated logarithm hold: for $0 \neq \Lambda < 1/2$, recalling γ_1 in (5), almost surely,

$$\limsup_{n\to\infty} \frac{W_n - \mathbb{E}[W_n]}{\gamma_1\sqrt{2n\log\log n}} = 1, \quad \liminf_{n\to\infty} \frac{W_n - \mathbb{E}[W_n]}{\gamma_1\sqrt{2n\log\log n}} = -1. \tag{10}$$

For $\Lambda = 1/2$, with γ_1 as in (5), almost surely,

$$\limsup_{n\to\infty} \frac{W_n - \mathbb{E}[W_n]}{\gamma_1\sqrt{2\log n\log\log\log n}} = 1, \quad \liminf_{n\to\infty} \frac{W_n - \mathbb{E}[W_n]}{\gamma_1\sqrt{2\log n\log\log\log n}} = -1. \tag{11}$$

Since we are focused on large-index and triangular urns in this paper, we do work out the proofs here.

The following two tables give a schematic summary of the main results of this work and [32]. To be more precise, our work adds the third order term in the third column of the first table and the second order term in both columns of the second table. Here, we write $X_n = Y_n + Z_n\mathcal{N}$ when $Z_n^{-1}(X_n - Y_n) \to \mathcal{N}$ in distribution. Further, we recall ζ and Q from (4) and Λ from (3).

$0 \neq \Lambda < 1/2$	$\Lambda = 1/2$	$1/2 < \Lambda < 1$
$W_n = \zeta n + \gamma_1\sqrt{n}\mathcal{N}$	$W_n = \zeta n + \gamma_1\sqrt{n\log n}\mathcal{N}$	$W_n = \zeta n + (\mathcal{W}_\infty + \vartheta)n^\Lambda + \alpha^{-1}\sqrt{n}\mathcal{N}$
γ_1 given in (5)	γ_1 given in (5)	ϑ given in (13), α given in (7)

TABLE 1. Behavior of W_n in small- and large-index balanced affine urns.

$\Lambda < 1$	$\Lambda = 1$
$W_n = Q^{-1}\mathfrak{W}_\infty n^\Lambda + (\beta Q)^{-1}\sqrt{\mathfrak{W}_\infty}n^{\Lambda/2}\mathcal{N}$	$W_n = Q^{-1}\mathfrak{W}_\infty n + (\beta Q)^{-1}\sqrt{\mathfrak{W}_\infty(T_0 - \mathfrak{W}_\infty)}\sqrt{n}\mathcal{N}$
β given in (9)	β given in (9)

TABLE 2. Behavior of W_n in triangular balanced affine urns.

[2] More precisely, $(\mathbb{E}[\mathfrak{W}_\infty^p])^{1/p} \sim cp^{1-\Lambda}$ as $p \to \infty$ where $c = c(a_0, \sigma)$.

Note that $\mathcal{W}_\infty, \mathfrak{W}_\infty$ as well as α, β, ϑ and Q depend on the initial configuration of the urn (W_0, B_0). Further, the distributions of the $\mathcal{W}_\infty$ and $\mathfrak{W}_\infty$ depend on the sampling scheme.

2.1. **Application I: Degrees in increasing trees.** We present an application in the case $m = 1$ for triangular urn schemes in the context of random linear recursive trees. These can be constructed as follows: at time $n = 1$, we start with a tree T_1 consisting of a single node. At time $n \geq 2$, given a tree T_{n-1} of size $n-1$, we choose a node v proportionally to $1 + \varrho d_v$, where d_v denote its out-degree (that is, its number of children) and $\varrho \in \mathbb{N}_0$. The tree T_n is then obtained by connecting an additional node to v. The most important models are $\varrho = 0$ (random recursive tree) and $\varrho = 1$ (plane-oriented recursive tree). By $D_n^{(i)}, n \geq 0$, we denote the out-degree of the i-th inserted node at time $n + i$. By construction, $W_n^{(i)} := \varrho D_n^{(i)} + 1$ is equal to the number of white balls in a two-color Pólya urn model with $m = 1, a_0 = \varrho, b_0 = 1, a_1 = 0, b_1 = \varrho + 1, W_0 = 1, B_0 = (\varrho + 1)(i - 1)$. In fact, $D_n^{(i)}$ counts the number of times we sample a white ball from the urn. From now on, assume $\varrho \geq 1$. Obviously, Theorem 1 *iii)* applies and we denote the martingale limit by $\mathfrak{W}_\infty^{(i)}$. An expression for the density of $\mathfrak{W}_\infty^{(i)}$ in terms of an infinite sum is given in [25, Theorem 9.1]. For $i = 1$, the term simplifies and the limit law is directly related to a distribution of Mittag-Leffler type. The cases $\alpha = 1, i \geq 1$ were studied in more detail by Peköz, Röllin and Ross [39] who give bounds on the convergence rates in the Kolmogorov distance and observe an interesting distributional identity for $\mathfrak{W}_\infty^{(i)}$ [39, Proposition 2.3]. In the next corollary, we only state the law of iterated logarithm, the novel contribution in our work (for $m = 1$). Here, as before, we write $\mathfrak{W}_n^{(i)}$ for the martingale corresponding to $W_n^{(i)}$ defined as in Theorem 3.

Corollary 1. *For $\varrho, i \geq 1$, almost surely,*

$$\limsup_{n\to\infty} \frac{n^{\varrho/(2(\varrho+1))}(\mathfrak{W}_n^{(i)} - \mathfrak{W}_\infty^{(i)})}{\sqrt{2\varrho \mathfrak{W}_\infty^{(i)} \log\log n}} = 1, \quad \liminf_{n\to\infty} \frac{n^{\varrho/(2(\varrho+1))}(\mathfrak{W}_n^{(i)} - \mathfrak{W}_\infty^{(i)})}{\sqrt{2\varrho \mathfrak{W}_\infty^{(i)} \log\log n}} = -1.$$

The case $\varrho = 0$ is substantially different. Here, $D_n^{(i)}$ can be expressed as the sum of independent Bernoulli random variables with success probabilities $1/j, j = i, \ldots, n+i-1$. Therefore, expectation and variance of $D_n^{(i)}$ are logarithmic and both central limit theorem as well as laws of the iterated logarithm are classical.

2.2. **Application II: Degrees in preferential attachment graphs.** Linear recursive trees are special instances of so-called *preferential attachment graphs* which play an important role in modeling scale-free networks with applications in sociology, neurology and computer science (e.g. the webgraph). This active research topic was initiated by the seminary work of Barabási and Albert [4]. Many authors studied dynamic networks in which new nodes are linked to several vertices in the graph. This leads to the construction of the following random circuit or directed acyclic (multi-)graph. At time $n = 1$, the graph G_1 consists of a single node. Given the directed graph G_{n-1} at time $n \geq 2$, we choose two nodes v, v' independently, each of which proportionally to $1 + \varrho d_v, \varrho \in \mathbb{N}_0$, where d_v counts the number of directed edges emanating from v. G_n is obtained by adding an additional node and directed links from both v and v' to the latter. For $\varrho = 0$, various quantities in the resulting network have been studied, compare Díaz et al. [11], Tsukiji and Xhafa [44], Devroye and Janson [10] for the height and Tsukiji and Mahmoud [45] for node degrees. As above, we denote by $D_n^{(i)}, n \geq 0$, the out-degree of the i-th inserted node at time $n + i$. Then, $W_n^{(i)} := \varrho D_n^{(i)} + 1$ is equal to the number of white balls in a two-color Pólya urn model with $m = 2, a_0 = 2\varrho, b_0 = 1, a_1 = \varrho, b_1 = \varrho + 1, a_2 = 0, b_2 = 2\varrho + 1, W_0 = 1, B_0 = (2\varrho + 1)(i - 1)$. The almost sure convergence in the next corollary follows immediately from Theorem 1 $iii)$ while the main theorems in this work give the Gaussian limit law, law of the iterated logarithm and the properties of the limiting random variable.

Corollary 2. *Let $\varrho, i \geq 1$ and $\Lambda = 2\varrho/(2\varrho+1), Q = (i-1)!/\Gamma(i-\Lambda)$. Then, almost surely and with convergence of all moments,*

$$\frac{Q \cdot W_n^{(i)}}{n^\Lambda} \to \mathfrak{W}_\infty^{(i)},$$

where the limit random variable has unit mean and its distribution admits a density and Subgaussian tails. The density can be chosen bounded for $\varrho = 1$. In distribution and with convergence of all moments,

$$n^{\Lambda/2}(n^{-\Lambda} Q \cdot W_n^{(i)} - \mathfrak{W}_\infty^{(i)}) \to \sqrt{\varrho \mathfrak{W}_\infty^{(i)}} \cdot \mathcal{N},$$

where $\mathfrak{W}_\infty^{(i)}$ and $\mathcal{N}$ are independent. Finally, almost surely,

$$\limsup_{n\to\infty} \frac{n^{\Lambda/2}(n^{-\Lambda} Q \cdot W_n^{(i)} - \mathfrak{W}_\infty^{(i)})}{\sqrt{2\varrho \mathfrak{W}_\infty^{(i)} \log\log n}} = 1, \quad \liminf_{n\to\infty} \frac{n^{\Lambda/2}(n^{-\Lambda} Q \cdot W_n^{(i)} - \mathfrak{W}_\infty^{(i)})}{\sqrt{2\varrho \mathfrak{W}_\infty^{(i)} \log\log n}} = -1.$$

Again, for $\varrho = 0$, node degrees grow logarithmically and one obtains similar results as in the case of random recursive trees, compare Mahmoud [36].

2.3. **Application III: Leaves in random circuits.** In this section, we consider the graph G_n constructed as above with $\varrho = 0$ and the modification that, in each step, the parent nodes v and v' are chosen without replacement. Denote by L_n the number of leaves (nodes with out-degree zero) in the graph. By observing that L_n coincides with the number of white balls in a two-color Pólya urn model (at time $n-2$) with $m = 2, a_0 = -1, b_0 = 2, a_1 = 0, b_1 = 1, a_2 = 1, b_2 = 0, W_0 = 1, B_0 = 1$, a Gaussian central limit theorem for L_n is proved in [45]. This also follows immediately from Theorem 1 *i)*. The next corollary states the accompanying law of the iterated logarithm and follows from (10).

Corollary 3. *Almost surely,*

$$\limsup_{n\to\infty} \frac{L_n - n/3}{\sqrt{n \log\log n}} = \frac{2\sqrt{10}}{15}, \quad \liminf_{n\to\infty} \frac{L_n - n/3}{\sqrt{n \log\log n}} = -\frac{2\sqrt{10}}{15}.$$

2.4. **Generalizations and outlook.** There are several possibilities to generalize and strengthen our results. First, distributional and almost sure convergence theorems remain valid for random initial configurations both on a conditioned and unconditioned level assuming tenability. Furthermore, our results can be extended to models with non-integer values of a_{m-1}, a_m and σ, such that $a_k \geq 0$, $0 \leq k \leq m$, assuming $W_0, B_0 \in [0, \infty)$ such that $W_0 + B_0 \geq m$ for model $\mathcal{M}$ and $W_0 + B_0 \geq 1$ for model $\mathcal{R}$ ensuring tenability. We expect that our methods also apply to the study of the one-dimensional limit random variables and their laws in linear affine balanced models with $r \geq 2$ colors. This will be content of future work.

3. Preliminaries

3.1. **Tenability.** Tenable urn schemes in the two-color case and $m = 1$ were classified in [2]. In multiple drawings schemes, sufficient conditions for tenability were formulated by Konzem and Mahmoud [28]. In the following lemma, we characterize tenability in the models considered in this work. Since the result is not at the core of our work, we defer its proof to the appendix.

Lemma 1. *Let* $\mathbf{M}$ *be the replacement matrix of a balanced affine urn process with $\Lambda \neq 0$ and initial configuration (W_0, B_0). Let $g_a = \mathbf{gcd}(a_0, \ldots, a_m), g_b = \mathbf{gcd}(b_0, \ldots, b_m)$ and, for $z \in \mathbb{Z}, y \in \mathbb{N}$, abbreviate $[z]_y := z \bmod y \in \{0, \ldots, y-1\}$.*

i) Under model $\mathcal{R}$, the scheme is tenable, if and only if, $a_1, \ldots, a_m, b_0, \ldots, b_{m-1} \geq 0$ and

$$a_0 \in \{z \in -\mathbb{N} : z | W_0, z | (a_{m-1} - a_m)\} \cup \mathbb{N}_0,$$
$$b_m \in \{z \in -\mathbb{N} : z | B_0, z | (a_{m-1} - a_m)\} \cup \mathbb{N}_0.$$

ii) Under model $\mathcal{M}$, the scheme is tenable, if and only if, $a_k \geq -(m-k), 1 \leq k \leq m$ and $b_k \geq -k, 0 \leq k \leq m-1$, and

$$a_0 \in [-m,\infty) \cup ([-m-g_a+1,-m) \cap \{z \in -\mathbb{N} : [W_0]_{g_a} \in \{[-z]_{g_a}, [-z+1]_{g_a}, \dots, [m+g_a-1]_{g_a}\})$$
$$b_m \in [-m,\infty) \cup ([-m-g_b+1,-m) \cap \{z \in -\mathbb{N} : [B_0]_{g_b} \in \{[-z]_{g_b}, [-z+1]_{g_b}, \dots, [m+g_b-1]_{g_b}\})$$

3.2. **Forward equations.** Following the notation introduced in [32], we write $\mathbf{1}_n(W^k B^{m-k})$ for the indicator function of the event that k white balls and $m-k$ black balls are drawn in the nth step. By the dynamics of the urn process, we have

$$W_n = W_{n-1} + \Delta_n, \qquad \Delta_n = \sum_{k=0}^{m} a_{m-k}\, \mathbf{1}_n(W^k B^{m-k}), \quad n \geq 1. \tag{12}$$

Thus,

$$p_{n;m,k} = \mathbb{P}\big(\Delta_n = a_{m-k} \,|\, \mathcal{F}_{n-1}\big) = \mathbb{E}[\mathbf{1}_n(W^k B^{m-k}) \,|\, \mathcal{F}_{n-1}],$$

where the conditional probabilities $p_{n;m,k}$ are given by

$$p_{n;m,k} = \begin{cases} \binom{W_{n-1}}{k}\binom{B_{n-1}}{m-k}/\binom{T_{n-1}}{m} & \text{for model } \mathcal{M}, \\ \binom{m}{k} W_{n-1}^k B_{n-1}^{m-k}/T_{n-1}^m & \text{for model } \mathcal{R}. \end{cases}$$

3.3. **The mean number of nodes.** We recall results on the mean number of white balls and a strong law of large numbers from [32].

Lemma 2 ([32])**.** *For both models $\mathcal{R}$ and $\mathcal{M}$ and $n \geq 1$, we have $\mathbb{E}[W_n] = \frac{a_m}{g_n}\sum_{j=1}^{n} g_j + W_0 \frac{1}{g_n}$ with g_n in* (6)*. For large-index and triangular urns with $\Lambda < 1$, it holds*

$$\begin{aligned} \mathbb{E}[W_n] &= \frac{a_m\left(n + \frac{T_0}{\sigma}\right)}{1-\Lambda} + \left(W_0 - \frac{\frac{a_m T_0}{\sigma}}{1-\Lambda}\right) \frac{\binom{n-1+\frac{T_0}{\sigma}+\Lambda}{n}}{\binom{n-1+\frac{T_0}{\sigma}}{n}} \\ &= \zeta n + \vartheta n^{\Lambda} + \mathcal{O}(1), \quad \vartheta = \left(W_0 - \frac{\frac{a_m T_0}{\sigma}}{1-\Lambda}\right) \frac{\Gamma(\frac{T_0}{\sigma})}{\Gamma(\frac{T_0}{\sigma}+\Lambda)}. \end{aligned} \tag{13}$$

For triangular urns with $\Lambda = 1$, we have

$$\mathbb{E}[W_n] = W_0 \frac{n\sigma + T_0}{T_0}.$$

For large-index urns, we have, almost surely,

$$\frac{W_n}{n} \to \zeta. \tag{14}$$

3.4. **Martingale tail sums and central limit theorems.** The following proposition on martingale tail sums is essentially a restatement of Theorem 1, Corollary 1 and Corollary 2 in Heyde [22]. The result concerning convergence of moments follows from Theorem 3.5 in Hall and Heyde [21], compare also Theorem 1 in Hall [20] for the case $\eta = 1$.

Proposition 1. *Let $Z_n, n \geq 0$, be a zero-mean, L_2-bounded martingale with respect to a filtration $\mathcal{G}_n, n \geq 0$. Let $X_n = Z_n - Z_{n-1}, n \geq 1, X_0 := 0$, and $s_n^2 = \sum_{i=n}^{\infty} \mathbb{E}[X_i^2]$. Denote Z the almost sure limit of Z_n. Assume that, for some non-zero and finite random variable η, we have, almost surely,*

$$s_n^{-2} \sum_{i=n}^{\infty} \mathbb{E}[X_i^2 | \mathcal{G}_{i-1}] \to \eta^2, \tag{15}$$

and, for all $\varepsilon > 0$,

$$s_n^{-2} \sum_{i=n}^{\infty} \mathbb{E}[X_i^2 \mathbf{1}_{\{|X_i| \geq \varepsilon s_n\}}] \to 0. \tag{16}$$

Then,

$$(\eta s_n)^{-1}(Z_n - Z) \to \mathcal{N}, \quad s_n^{-1}(Z_n - Z) \to \eta' \mathcal{N}, \tag{17}$$

in distribution, where η' and $\mathcal{N}$ are independent and η' is distributed like η. If

L1. $\sum_{i=1}^{\infty} s_i^{-1}\mathbb{E}[|X_i|\mathbf{1}_{\{|X_i|\geq \varepsilon s_i\}}] < \infty$ *for all* $\varepsilon > 0$*,*
L2. $\sum_{i=1}^{\infty} s_i^{-4}\mathbb{E}[X_i^4] < \infty$*,*

then, almost surely,

$$\limsup_{n\to\infty} \frac{Z_n - Z}{\eta s_n\sqrt{2\log\log s_n^{-1}}} = 1, \quad \liminf_{n\to\infty} \frac{Z_n - Z}{\eta s_n\sqrt{2\log\log s_n^{-1}}} = -1.$$

Finally, if for all $p \in \mathbb{N}$ *sufficiently large,*

P1. Z_n *is bounded in* L_p*,*
P2. $s_n^{-2p}\sum_{i=n}^{\infty}\mathbb{E}[X_i^{2p}] \to 0$*,*
P3. $s_n^{-2}\sum_{i=n}^{\infty}\mathbb{E}[X_i^2|\mathcal{G}_{i-1}]$ *is bounded in* L_p*,*

then the second convergence in (17) *is with respect to all moments.*

3.5. **Martingale concentration inequalities.** In order to show that the limits $\mathcal{W}_\infty$ and $\mathfrak{W}_\infty$ have light tails, we need the following two tail bounds for martingales. The first is a well-known variant of Azuma's inequality.

Proposition 2. *Let* $M_0, M_1, \ldots$ *be a martingale with respect to a filtration* $\mathcal{G}_0, \mathcal{G}_1, \ldots$ *Assume that, for all* $n \geq 1$*, there exist constants* $c_n \geq 0$ *and* $\mathcal{G}_{n-1}$ *measurable random variables* Z_n*, such that, almost surely,* $0 \leq M_n - M_{n-1} - Z_n \leq c_n$*. Then,*

$$\mathbb{P}(|M_n - M_0| \geq t) \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\right), \quad t > 0.$$

The next proposition is a variant of Bennett's inequality for martingales which improves on the previous proposition when the conditional variances of the martingale differences are significantly smaller than their essential suprema.

Proposition 3 (Chung and Lu [9], Theorem 7.3)**.** *Let* $M_0, M_1, \ldots$ *be a martingale with respect to a filtration* $\mathcal{G}_0, \mathcal{G}_1, \ldots$ *where* M_0 *is almost surely constant. Assume that there exists a constant* $M > 0$ *and, for all* $n \geq 1$*, non-negative constants* σ_n, ϕ_n *such that, almost surely,* $M_n - M_{n-1} \leq M$ *and* $\mathbb{E}[(M_n - M_{n-1})^2|\mathcal{G}_{n-1}] \leq \sigma_n + \phi_n M_{n-1}$*. Then,*

$$\mathbb{P}(M_n - M_0 \geq t) \leq \exp\left(-\frac{t^2}{2(Mt/3 + \sum_{i=1}^{n}\sigma_i + (M_0 + t)\sum_{i=1}^{n}\phi_i)}\right), \quad t > 0. \tag{18}$$

4. Proofs of the main results

We start be recalling that the process $\mathcal{W}_n$ and $\mathfrak{W}_n$ are martingales [32, Proposition 4]. To unify the notation, let Y_n and e_n be defined as

$$Y_n = g_n(W_n - e_n), \quad \text{with} \quad e_n = \begin{cases} 0 & \text{for triangular urn models,} \\ \mathbb{E}[W_n] & \text{for large-index urn models,} \end{cases}$$

such that

$$Y_n = \begin{cases} \mathfrak{W}_n & \text{for triangular urn models,} \\ \mathcal{W}_n & \text{for large-index urn models.} \end{cases}$$

In the remaining of the paper, we denote by X_n the martingale difference

$$X_n = Y_n - Y_{n-1}, \quad n \geq 1.$$

4.1. **Martingale differences: bounds and moments.** Aiming at applications of Proposition 1 we investigate the martingale differences $X_n, n \geq 1$. By the simple Lemma 3 in [32] which applies to all our models, there exists a deterministic constant K such that, for all $n \geq 1$, we have

$$|X_n| \leq K n^{-\Lambda}. \tag{19}$$

It turns out that this rather trivial bound is of the right order for large-index urns with $1/2 < \Lambda < 1$ and triangular urns with $\Lambda = 1$. Therefore (19) is sufficient to verify conditions **L2**, **P2** and **P3** in Proposition 1. In the case of triangular urns, condition **L2** can be checked with the help of (19) for $\Lambda > 1/2$. It is only the case of triangular urns with $\Lambda \leq 1/2$ where more precise estimates are required. For the sake of completeness, we give the first order terms of $\mathbb{E}[X_n^4]$ in all cases below in Lemma 4. Further, (19) also settles (16) and condition **L2** as will become clear in the proof of the main theorems below.

Our starting point to compute second and fourth moments is the following observation which can be verified by means of direct computations:

$$X_{n+1} = g_{n+1}\big(\hat{Y}_n + \Delta_{n+1} - a_m\big), \quad \hat{Y}_n = \Big(\frac{1}{g_n} - \frac{1}{g_{n+1}}\Big)Y_n + e_n - e_{n+1} + a_m,$$

with Δ_n as given in (12). By the forward equation (12), it follows that the conditional distribution of X_{n+1} given $\mathcal{F}_n$ is

$$\mathbb{P}\left(X_{n+1} = g_{n+1}\left(\hat{Y}_n + k\frac{\sigma}{m}\Lambda\right)\Big|\mathcal{F}_n\right) = p_{n+1;m,k}, \quad 0 \leq k \leq m. \tag{20}$$

In the following we collect the asymptotic expansions of the conditional and unconditional expected value of the second moment.

Lemma 3. *We have the following asymptotic statements for both model $\mathcal{M}$ and model $\mathcal{R}$:*

- *For triangular urn models with $\Lambda = 1$:*

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] \sim \frac{T_0^2}{mn^2}\left(\frac{\mathfrak{W}_\infty}{T_0}\Big(1 - \frac{\mathfrak{W}_\infty}{T_0}\Big)\right), \quad \mathbb{E}[X_{n+1}^2] \sim \frac{T_0^2}{mn^2}\cdot\mathbb{E}\left[\frac{\mathfrak{W}_\infty}{T_0}\Big(1 - \frac{\mathfrak{W}_\infty}{T_0}\Big)\right].$$

  and

$$s_n^2 \sim \frac{T_0^2}{mn}\cdot\mathbb{E}\left[\frac{\mathfrak{W}_\infty}{T_0}\Big(1 - \frac{\mathfrak{W}_\infty}{T_0}\Big)\right].$$

- *For triangular urn models with $\Lambda < 1$:*

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] \sim n^{-\Lambda-1}\frac{\sigma Q\Lambda^2}{m}\mathfrak{W}_\infty, \quad \mathbb{E}[X_{n+1}^2] \sim n^{-\Lambda-1}\frac{\sigma Q\Lambda^2 W_0}{m}.$$

  and

$$s_n^2 \sim n^{-\Lambda}a_{m-1}\Lambda Q W_0.$$

- *For large-index urns:*

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] \sim n^{-2\Lambda}\frac{a_m b_0 Q^2\Lambda^2}{m(1-\Lambda)^2}, \quad \mathbb{E}[X_{n+1}^2] \sim n^{-2\Lambda}\frac{a_m b_0 Q^2\Lambda^2}{m(1-\Lambda)^2},$$

  and

$$s_n^2 \sim n^{-2\Lambda+1}\frac{a_m b_0 Q^2\Lambda^2}{m(2\Lambda-1)(1-\Lambda)^2}.$$

Proof. First, we derive an exact representation of the second moment in terms of $\hat{Y}_n$ and Y_n. By (20), almost surely,

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] = g_{n+1}^2\sum_{k=0}^{m}\left(\hat{Y}_n + k\frac{\sigma}{m}\Lambda\right)^2 p_{n+1;m,k} = g_{n+1}^2\sum_{k=0}^{m}\left(\hat{Y}_n^2 + 2\hat{Y}_n\frac{\sigma}{m}\Lambda k + \frac{\sigma^2\Lambda^2}{m^2}k^2\right)p_{n+1;m,k}.$$

The sums are readily evaluated using the basic properties of the binomial and the hypergeometric distributions. We obtain the expression

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] = g_{n+1}^2\left[\hat{Y}_n^2 + 2\hat{Y}_n\left(\frac{Y_n}{g_n} + e_n\right)\frac{\sigma\Lambda}{T_n}\right] \tag{21}$$

$$+ g_{n+1}^2 \frac{\sigma^2\Lambda^2}{mT_n}\left(\frac{Y_n}{g_n}+e_n\right) \times \begin{cases} \left(\frac{m-1}{T_n-1}\left(\frac{Y_n}{g_n}+e_n-1\right)+1\right) & \text{for model } \mathcal{M}, \\ \left(\frac{m-1}{T_n}\left(\frac{Y_n}{g_n}+e_n\right)+1\right) & \text{for model } \mathcal{R}. \end{cases} \tag{22}$$

We continue with the case of triangular urn models with $a_m = 0$, $e_n = 0$. From the last display, it follows

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] = g_{n+1}^2\left[\hat{Y}_n^2 + 2\hat{Y}_n\frac{Y_n}{g_n}\frac{\sigma\Lambda}{T_n}\right] + g_{n+1}^2\frac{\sigma^2\Lambda^2}{mT_n}\frac{Y_n}{g_n} \times \begin{cases} \left(\frac{m-1}{T_n-1}\left(\frac{Y_n}{g_n}-1\right)+1\right) & \text{for model } \mathcal{M}, \\ \left(\frac{m-1}{T_n}\frac{Y_n}{g_n}+1\right) & \text{for model } \mathcal{R}. \end{cases} \tag{23}$$

Assume first that $\Lambda = 1$, such that $\hat{Y}_n = -\frac{\sigma}{T_0}Y_n$. We get

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] = -g_{n+1}^2\frac{\sigma^2}{T_0^2}Y_n^2 + g_{n+1}^2\frac{\sigma^2}{mT_0}Y_n \times \begin{cases} \left(\frac{m-1}{T_n-1}\left(\frac{Y_nT_n}{T_0}-1\right)+1\right) & \text{for model } \mathcal{M}, \\ \left(\frac{m-1}{T_n}\frac{Y_nT_n}{T_0}+1\right) & \text{for model } \mathcal{R}. \end{cases}$$

Since Y_n converges with respect to all moments [33], for both sampling schemes, we deduce

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] \sim g_{n+1}^2\Big[-\frac{\sigma^2}{T_0^2}\mathfrak{W}_\infty^2 + \frac{\sigma^2}{mT_0}\mathfrak{W}_\infty\Big((m-1)\frac{\mathfrak{W}_\infty}{T_0}+1\Big)\Big], \tag{24}$$

which gives the stated result recalling (6). For $\Lambda < 1$, first note that

$$\frac{g_{n+1}}{g_n} = 1 - \frac{\Lambda}{n + T_0/\sigma + \Lambda}. \tag{25}$$

It is easy to identify the leading term in the expansion (23) which suffices to prove the statement. However, for later purposes, we state a more precise result. Applying (25) to the summands in (23), we have

$$\mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] = \Lambda a_{m-1}g_{n+1}\frac{Y_n}{n} - \frac{Y_n^2}{mn^2} + \mathcal{O}(n^{-1-2\Lambda}), \tag{26}$$

the $\mathcal{O}$-term being deterministic. By (6), as $n \to \infty$, the first summand dominates. For large-index urns, since (25) remains valid in this model, we have $\hat{Y}_n \sim e_n - e_{n+1} + a_m$. Using (13), we observe that

$$\hat{Y}_n \sim -\frac{a_m\Lambda}{1-\Lambda}.$$

From the explicit representation (22) we collect the dominant contributions and obtain the model-independent expansion

$$\begin{aligned} \mathbb{E}[X_{n+1}^2 \,|\, \mathcal{F}_n] &\sim g_{n+1}^2\Big[\frac{\Lambda^2a_m^2}{(1-\Lambda)^2} - 2\frac{\Lambda^2a_m^2}{(1-\Lambda)^2} + \frac{\sigma^2\Lambda^2}{m\sigma}\cdot\frac{a_m}{1-\Lambda}\Big(1+\frac{m-1}{\sigma}\cdot\frac{a_m}{1-\Lambda}\Big)\Big] \\ &\sim Q^2n^{-2\Lambda}\frac{-a_m\Lambda^2(a_m+\Lambda\sigma-\sigma)}{m(1-\Lambda)^2} = n^{-2\Lambda}\frac{a_mb_0Q^2\Lambda^2}{m(1-\Lambda)^2}. \end{aligned} \tag{27}$$

Note that we have used (14) here. Since the convergence of Y_n is with respect to all moments [33, Theorem 1], the equivalences in (24) and (27) as well as the expansion (26) also hold in mean. The expansions of s_n^2 are readily obtained using the Euler-MacLaurin summation formula. □

In the following lemma, denote by $r(m,p)$ the fourth central moment of the binomial distribution $\mathbf{bin}(m,p)$.

Lemma 4. *For $n \to \infty$ the fourth moment $\mathbb{E}[X_{n+1}^4]$ of the martingale difference satisfies for both model $\mathcal{M}$ and model $\mathcal{R}$:*

- *For triangular urn models with $\Lambda = 1$:*

$$\mathbb{E}[X_{n+1}^4] \sim \frac{T_0^4}{m^4}\mathbb{E}\left[r\left(m, \frac{\mathfrak{W}_\infty}{T_0}\right)\right]n^{-4}.$$

- *For triangular urn models with $\Lambda < 1$:*

$$\mathbb{E}[X_{n+1}^4] \sim \frac{\sigma^3Q^3W_0}{m^3}n^{-3\Lambda-1}.$$

- *For large-index urns:*

$$\mathbb{E}[X_{n+1}^4] \sim \left(\frac{\sigma\Lambda}{m}\right)^4 r\left(m, \frac{\zeta}{\sigma}\right) n^{-4\Lambda}.$$

Proof. By (20), in model $\mathcal{R}$,

$$\mathbb{E}[X_{n+1}^4] = g_{n+1}^4 \mathbb{E}\left[\left(\hat{Y}_n + \frac{\sigma\Lambda}{m}\text{Bin}\left(m, \frac{W_n}{T_n}\right)\right)^4\right], \tag{28}$$

where $\text{Bin}(m,p)$ denotes a random variable with distribution $\mathbf{bin(m,p)}$. In the triangular case with $\Lambda = 1$, since Y_n converges with respect to all moments,

$$\mathbb{E}[X_{n+1}^4] \sim g_{n+1}^4 \mathbb{E}\left[\left(-\frac{\sigma}{T_0}\mathfrak{W}_\infty + \frac{\sigma}{m}\text{Bin}\left(m, \frac{\mathfrak{W}_\infty}{T_0}\right)\right)^4\right].$$

The assertion now follows from (6). For large-index urns, by the same arguments,

$$\mathbb{E}[X_{n+1}^4] \sim g_{n+1}^4 \mathbb{E}\left[\left(-\zeta\Lambda + \frac{\sigma\Lambda}{m}\text{Bin}\left(m, \frac{\zeta}{\sigma}\right)\right)^4\right].$$

Since the binomial distribution $\mathbf{bin}(m,p)$ is the distributional limit of the hypergeometric distribution $\mathbf{hyp}(a(n), b(n), m)$ as $b(n)/a(n) \to p$, the same results hold under model $\mathcal{M}$. Note however, that the limiting random variable $\mathfrak{W}_\infty$ depends on the sampling scheme. For triangular urns with $\Lambda < 1$, one needs to be more precise. First, applying the binomial theorem to (28) gives

$$g_{n+1}^{-4}\mathbb{E}[X_{n+1}^4] = \sum_{k=0}^{4}\binom{4}{k}\left(\frac{\sigma\Lambda}{m}\right)^{4-k}\mathbb{E}\left[\hat{Y}_n^k\left(\text{Bin}\left(m, \frac{W_n}{T_n}\right)\right)^{4-k}\right].$$

Upon bounding the Binomial random variable from above by m, it follows that the summands $k = 2, 3, 4$ are of the order at most $n^{2\Lambda-2}$ and turn out to be asymptotically negligible. Regarding the summand $k = 1$, note that

$$\mathbb{E}\left[n^{1-\Lambda}\hat{Y}_n\left(\text{Bin}\left(m, \frac{W_n}{T_n}\right)\right)^3\right] \to 0, \quad n \to \infty.$$

This follows by the theorem of dominated convergence since it holds in probability for the integrand which it is moreover bounded from above by $m^2 a_{m-1}$. Finally, for the last summand $k = 0$, we compute

$$g_{n+1}^4 \mathbb{E}\left[\frac{\sigma^4}{\Lambda^4 m^4}\left(\text{Bin}\left(m, \frac{W_n}{T_n}\right)\right)^4\right] \sim \frac{\sigma^3 Q^3 W_0}{m^3} n^{-3\Lambda-1}.$$

Here, we have used that $\mathbb{E}\left[\text{Bin}(m,p)^4\right] \sim mp + \mathcal{O}(p^2)$ as $p \to 0$. This finishes the proof for model $\mathcal{R}$. For model $\mathcal{M}$, again by approximating the hypergeometric distribution by the binomial distribution, we obtain the analogous result. $\square$

In the case of triangular urns with $\Lambda = 1$ in Lemma 3 and 4 we have implicitly assumed that $0 < \mathfrak{W}_\infty < T_0$ almost surely. This will be justified later when we show that $\mathfrak{W}_\infty$ has a density on $[0, T_0]$ without relying on any results in the lemmas. Analogously, we will show that $\mathfrak{W}_\infty > 0$ almost surely for $\Lambda < 1$ needed in Lemma 3.

4.2. **Proofs of Theorems 2 and 3.** We start with the central limit theorems and the laws of the iterated logarithm relying on Proposition 1 postponing the verification that the martingale limits have non-atomic distributions. First of all, (15) with η given as in the theorems as well as condition **L2** can be checked directly with the help of Lemmas 3 and 4. Using the expansion of s_n in Lemma 3 and the bound (19), it is easy to see that, in all three urn models and for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have $|X_n| < \varepsilon s_n$. This verifies conditions (16) and **L1**. It remains to check conditions **P1**, **P2**, **P3** in Theorem 2 (and in Theorem 3 for $\Lambda > 1/2$). The moment convergence **P1** was proved in [33, Theorem 1], it also follows from the fact that the limiting random variables have exponentially small tails as shown below. **P2** immediately follows from (19). Similarly, **P3** follows by an application of Minkowski's inequality.

We move on to the tail bounds on the limiting random variables. To this end, note that

$$X_n - (g_n - g_{n-1})W_{n-1} = g_n(W_n - W_{n-1}) - a_m g_{n-1}.$$

Hence, choosing $C > 0$ such that $g_n \leq C(n+1)^{-\Lambda}$ and denoting $q = \max |a_k|$, by Proposition 2,

$$\mathbb{P}(|\mathcal{W}_n| \geq t) \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^n q^2(g_i + g_{i-1})^2}\right) \leq 2\exp\left(-\frac{t^2}{2q^2C^2\sum_{i=1}^n i^{-2\alpha}}\right).$$

Thus, $\mathcal{W}_\infty$ has Subgaussian tails. The claim follows analogously in the case of triangular urns for $\Lambda > 1/2$. For triangular urns with $\Lambda \leq 1/2$, by Proposition 3, using (19), in order to show a bound of the form (18) for $\mathfrak{W}_n$, it is enough to verify that, almost surely,

$$\mathbb{E}[X_{n+1}^2|\mathcal{F}_n] \leq \sigma_n + \phi_n \mathfrak{W}_n, \quad n \geq 0, \tag{29}$$

with deterministic, non-negative and summable sequences σ_n, ϕ_n. It is here where we need the full strength of expansion (26). By the triangle inequality, upon bounding one of the factors Y_n in the second term from above by $g_n T_n$, there exists a deterministic number $C > 0$, such that, for all $n \geq 0$,

$$\mathbb{E}[X_{n+1}^2|\mathcal{F}_n] \leq Q(\Lambda a_{m-1} + \sigma)\mathfrak{W}_n n^{-1-\Lambda} + Cn^{-1-\Lambda}.$$

Thus, (29) is satisfied with $\phi_n = \mathcal{O}(n^{-1-\Lambda})$ and $\sigma_n = \mathcal{O}(n^{-1-\Lambda})$.

In order to show the existence of a density for $\mathfrak{W}_\infty$, we roughly follow the ideas in [7]. By the postponed Lemma 5, we need to show that

$$p_n := \max_{0 \leq k \leq m(n+1)} \mathbb{P}(W_n = W_0 + ka_{m-1}) = \mathcal{O}(n^{-\Lambda}). \tag{30}$$

Let

$$s^*(k,n) := \sum_{i \in S_n} \mathbb{P}(W_{n+1} = W_0 + ka_{m-1}|W_n = W_0 + (k-i)a_{m-1}),$$

where S_n denotes the set of integers $0 \leq i \leq m$ with $\mathbb{P}(W_n = W_0 + (k-i)a_{m-1}) > 0$. Decomposing with respect to the position of the Markov chain W_n at time $n-1$ gives rise to

$$p_n \leq \max_{0 \leq k \leq m(n+1)} s^*(k,n)p_{n-1} \leq \prod_{i=1}^n \max_{0 \leq k \leq m(i+1)} s^*(k,i) \leq \exp\left(-\sum_{i=1}^n \left(1 - \max_{0 \leq k \leq m(i+1)} s^*(k,i)\right)\right).$$

For a set $A \subseteq [0, n+1] := \{0, 1, \ldots, n, n+1\}$, let $s(A) := \sup_{k \in A} s^*(k,n)$. Then, (30) follows if we can show that,

$$s([0, n+1]) = 1 - \frac{\Lambda}{n} + \mathcal{O}(n^{-2}). \tag{31}$$

Note that,

$$s(\{0\}) = s^*(0,n) = \left(1 - \frac{W_0}{T_n}\right)^m = 1 - \frac{mW_0}{n\sigma} + \mathcal{O}(n^{-2}), \tag{32}$$

and

$$s(\{n+1\}) = s^*(n+1,n) = \left(1 - \frac{B_0}{T_n}\right)^m = 1 - \frac{mB_0}{n\sigma} + \mathcal{O}(n^{-2}), \tag{33}$$

Next, for $i \in S_n$,

$$\begin{aligned}\mathbb{P}(W_{n+1} = W_0 + a_{m-1}k|W_n = W_0 + a_{m-1}(k-i))\\ = \frac{1}{T_n^m}\binom{m}{i}(W_0 + (k-i)a_{m-1})^i (T_n - W_0 - (k-i)a_{m-1})^{m-i}.\end{aligned}$$

We start with the case $\Lambda < 1$. Since $\sigma > ma_{m-1}$, there exists n_0 such that, for all $n \geq n_0$, we have $T_n - W_0 - (k-i)a_{m-1} \geq 0$. For these n, we can bound $s^*(k,n) \leq s(k,n)$ with

$$s(k,n) = \sum_{i=0}^{\min(k,m)} \frac{1}{T_n^m}\binom{m}{i}(W_0 + (k-i)a_{m-1})^i (T_n - W_0 - (k-i)a_{m-1})^{m-i}. \tag{34}$$

Direct computations show that

$$s([1,m]) = 1 - \frac{\Lambda}{n} + \mathcal{O}(n^{-2}). \tag{35}$$

For $k \geq m$ the same expansion is essentially given in Lemma 4.2 in [8]; however, the arguments there are incomplete. By an application of the binomial theorem, we arrive at an expression computed on page 1182 in [8]: for $k \geq m$,

$$s(k,n) = \frac{1}{T_n^m} \sum_{\ell=0}^{m} a_{m-1}^{m-\ell} T_n^\ell \binom{m}{\ell} \sum_{i=0}^{m-\ell} \binom{m-\ell}{i} (-1)^{m-\ell-i} (W_0/a_{m-1} + k - i)^{m-\ell}.$$

From here, we use the following identity, compare equation (5.42) in Graham, Knuth and Patashnik [18],

$$\sum_{i\geq 0} \binom{j}{i} (-1)^i (d_0 + d_1 i + \cdots + d_j i^j) = (-1)^j j! d_j, \quad j \in \mathbb{N}_0, \tag{36}$$

with arbitrary coefficients d_ℓ, in order to evaluate the inner sum

$$\sum_{i=0}^{m-\ell} \binom{m-\ell}{i} (-1)^{m-\ell-i} (W_0/a_{m-1} + k - i)^{m-\ell} = (-1)^{m-\ell}(m-\ell)!.$$

Consequently, we obtain

$$s(k,n) = \frac{m!}{T_n^m} \sum_{\ell=0}^{m} (-1)^{m-\ell} a_{m-1}^{m-\ell} \frac{T_n^\ell}{\ell!}.$$

In particular, we make the crucial observation that, for $k \geq m$, $s(k,n)$ is independent of k. (Note that it is also independent of W_0.) Hence,

$$s([m, m(n+1)]) = 1 - \frac{m a_{m-1}}{T_n} + \mathcal{O}(n^{-2}) = 1 - \frac{\Lambda}{n} + \mathcal{O}(n^{-2}). \tag{37}$$

Combining the last display, (32) and (35), since $W_0 \geq a_{m-1}$, we have proved (31). For $\Lambda = 1$, we can use the bound (34) only for $k \leq mn + B_0/a_{m-1}$. Thus, $s([m, mn + \lfloor B_0/a_{m-1} \rfloor]) = 1 - \frac{\Lambda}{n} + \mathcal{O}(n^{-2})$. For $mn + \lfloor B_0/a_{m-1} \rfloor < k \leq m(n+1) - 1$, we can bound

$$s^*(k,n) \leq \sum_{i=k-mn}^{m} \frac{1}{T_n^m} \binom{m}{i} (W_0 + (k-i)a_{m-1})^i (T_n - W_0 - (k-i)a_{m-1})^{m-i}.$$

Again, by direct computation one can check that only the summands $i = m, m-1$ are of relevance and lead to a bound of the right order. (31) now follows as in the case $\Lambda < 1$ where we additionally need (33) and $B_0 \geq a_{m-1}$.

We move on to the case $W_0 < a_{m-1}, \Lambda < 1$, and use the notation $\mathfrak{W}_\infty(w,b)$ for the martingale limit when the process is started with w white and b black balls. Let $\ell > 0$ and $j \geq a_{m-1}$ such that $\mathbb{P}(W_\ell = j) > 0$. Conditioned on the event $\{W_\ell = j\}$, by the Markov property of W_n, the limit $\mathfrak{W}_\infty(W_0, B_0)$ is distributed like $\mathfrak{W}_\infty(j, T_\ell - j)$ (modulo a deterministic factor due to the normalization). Therefore, it has a bounded density. Hence, the distribution of $\mathfrak{W}_\infty(W_0, B_0)$ (now, unconditionally) admits a (possibly unbounded) density if $W_\ell \to \infty$ almost surely as $\ell \to \infty$. This follows from the central limit theorem noting that, in probability, W_ℓ can be bounded from below by the sum of independent Bernoulli variables with success probabilities $W_0/T_i, 1 \leq i \leq n$. Alternatively, this also follows from an application of a conditional version of the second Borel-Cantelli Lemma as worked out in [7]. For $W_0, B_0 < a_{m-1}$ and $\Lambda = 1$, the proof runs along the same lines.

For model $\mathcal{M}$, similar arguments apply and we only consider the details in the main regime where $m \leq k \leq m(n+1)$ assuming for simplicity that $\Lambda < 1$. Here, we improve upon an argument from [7]: similarly to (34), for all n sufficiently large, we define

$$s(k,n) = \sum_{i=0}^{m} \frac{\binom{W_0 + (k-i)a_{m-1}}{i} \binom{T_n - W_0 - (k-i)a_{m-1}}{m-i}}{\binom{T_n}{m}}$$

$$= \frac{1}{T_n^{\underline{m}}} \sum_{i=0}^{m} \binom{m}{i} \big(T_n - W_0 - (k-i)a_{m-1}\big)^{\underline{m-i}} \big(W_0 + (k-i)a_{m-1}\big)^{\underline{i}}.$$

By the binomial theorem for the falling factorials, we obtain after a change of summation

$$s(k,n) = \frac{1}{T_n^{\underline{m}}} \sum_{\ell=0}^{m} \binom{m}{\ell} T_n^{\underline{\ell}} \sum_{i=0}^{m-\ell} (-1)^{m-i-\ell} \big(-W_0 - (k-i)a_{m-1}\big)^{\underline{m-i-\ell}} \big(W_0 + (k-i)a_{m-1}\big)^{\underline{i}}.$$

The product of the falling factorials is a polynomial in the variable i of degree $m-\ell$ with leading coefficient $(-1)^{m-\ell}a_{m-1}^{m-\ell}$; the concrete values of the other coefficients are of no importance. By identity (36), we obtain

$$\sum_{i=0}^{m-\ell} (-1)^{m-i-\ell} \big(-W_0 - (k-i)a_{m-1}\big)^{\underline{m-i-\ell}} \big(W_0 + (k-i)a_{m-1}\big)^{\underline{i}} = (-1)^{m-\ell} a_{m-1}^{m-\ell} (m-\ell)!,$$

such that

$$s(k,n) = \frac{m!}{T_n^{\underline{m}}} \sum_{\ell=0}^{m} (-1)^{m-\ell} a_{m-1}^{m-\ell} \frac{T_n^{\underline{\ell}}}{\ell!}.$$

This directly leads to the desired expansion (37) for model $\mathcal{M}$.

Considering large-index urns, by the same argument applied to $W_n - \mathbb{E}[W_n]$, the existence of a bounded density for $\mathcal{W}_\infty$ follows if, uniformly in $0 \le k \le m(n+1)$, as $n \to \infty$,

$$\sum_{i=0}^{m} \mathbb{P}(W_{n+1} = W_0 + (n+1)a_m + hk | W_n = W_0 + na_m + h(k-i)) = 1 - \frac{\Lambda}{n} + \mathcal{O}(n^{-2}),$$

where we abbreviated $h = a_{m-1} - a_m$. The latter follows by the same calculation as above. This finishes the proof.

Lemma 5. *Let X_n be a sequence of random variables and g_n a real-valued sequence such that, for all $n \ge 1$, the difference $X_n - g_n$ is integer-valued. Assume that, for some $\alpha > 0$ and some finite random variable X, we have $n^{-\alpha}X_n \to X$ in distribution. If,*

$$K := \liminf_{n\to\infty} n^{\alpha} \max_{m\in\mathbb{Z}} \mathbb{P}(X_n - g_n = m) < \infty,$$

then X admits a density on $(-\infty, \infty)$ which can be bounded uniformly by K.

Proof. For $-\infty < a < b < \infty$, by the Portmanteau Lemma,

$$\begin{aligned}\mathbb{P}(a < X < b) &\le \liminf_{n\to\infty} \mathbb{P}(an^{\alpha} < X_n < bn^{\alpha}) \\ &\le \liminf_{n\to\infty} \max_{m\in\mathbb{Z}} \mathbb{P}(X_n - g_n = m)(b-a)(n^{\alpha}+2) = K(b-a).\end{aligned}$$

Thus, the distribution function of X is Lipschitz and therefore absolutely continuous. For any density f, the last display implies that $f \le K$ Lebesgue almost everywhere. The claim follows by modifying f on a null-set if necessary. ☐

5. Appendix

Proof of Lemma 1. A scheme is tenable if and only if, almost surely, for all $n \ge 1$, both $W_{n-1} \ge -a_{m-R_n}$ and $B_n \ge -b_{R_n}$, where R_n denotes the number of white balls in the sample obtained in step n. Hence, under model $\mathcal{R}$, the process is tenable if all coefficients of $\mathbf{M}$ are non-negative. Similarly, tenability follows in model $\mathcal{M}$ if $a_k \ge -(m-k), b_k \ge -k$ for all $0 \le k \le m$. For a tenable urn scheme, we define $\mathcal{R}$ as the range of the Markov chain W_n, that is, the set of integers $\ell \in \mathbb{N}_0$ with $\mathbb{P}(W_n = \ell) > 0$ for some $n \ge 0$.

We consider model $\mathcal{R}$, assume that $W_0, B_0 \ge 1, a_j < 0$ for some $0 \le j \le m$ and tenability. Then, $a_i < a_j$ for $0 \le i \le j-1$. If $j > 0$, then, at time $t = \lceil -W_0/a_j \rceil - 1$, we have $0 < W_t \le -a_j$ with positive probability. This contradicts tenability. Hence, $j = 0$. It is clear that $a_0 | z$ for all $z \in \mathcal{R}$. In particular, $a_0 | W_0$ and $a_0 | (W_0 + (a_{m-1} - a_m))$, hence $a_0 | (a_{m-1} - a_m)$. The same arguments apply for $b_0, \ldots, b_m$. The cases $W_0 = 0$ or $B_0 = 0$ can be treated analogously. Finally, it is obvious that the urn process is tenable if a_0, b_m satisfy the conditions stated in the theorem and all remaining coefficients are non-negative. This finishes the proof of *i)*.

We move on to model $\mathcal{M}$. If $a_{m-1} - a_m \geq -1$, then, since $a_m \geq 0$, we have $a_0 \geq -m$. Hence, the only interesting case is when $h := a_{m-1} - a_m \leq -2$ and $a_k < -(m-k)$ for some $0 \leq k \leq m$. Assume the scheme is tenable and let j be maximal with $a_j < -(m-j)$. If $j \geq 1$, then $a_1 < -m+1$ and $m-h-1 < -a_0$. As $\Delta < 0$, the urn is of small index. We have $W_n, B_n \to \infty$ almost surely. Given a sufficiently large number of white and black balls in the urn, upon first drawing ℓ_1 samples containing m white balls and then ℓ_2 samples containing $m-1$ white balls, we remove $-\ell_1 a_0 - \ell_2 a_1$ white balls from the urn. Therefore, there exists $r \in \mathcal{R}$ with $m \leq r \leq m+h-1$. But then $r < -a_0$ violating tenability. It follows that $j = 0$, that is, $a_0 < -m$ and $a_k \geq -(m-k)$ for $k = 1, \dots, m$. Obviously, $\mathcal{R} \cap [m, \infty] \subseteq (W_0 + g_a\mathbb{Z}) \cap [m, \infty]$. For $r \geq m$ with $r = W_0 + cg_a, c \in \mathbb{Z}$, we have $r \in \mathcal{R}$ if $r - da_0 \in \mathcal{R}$ for some $d \in \mathbb{N}$. Again, since the urn is of small index, $r - da_0 \in \mathcal{R}$ holds for all d sufficiently large. Hence, $\mathcal{R} \cap [m, \infty] = (W_0 + g_a\mathbb{Z}) \cap [m, \infty]$. Obviously, we must have $g_a \geq -a_0 - m + 1$. In that case, tenability implies that $[W_0]_{g_a} \notin \{[m]_{g_a}, \dots, [-a_0-1]_{g_a}\}$. On the other hand, if $a_0 < -m$ and $[W_0]_{g_a} \notin \{[m]_{g_a}, \dots, [-a_0-1]_{g_a}\}$, then $[m, -a_0-1] \cap \mathcal{R} = \emptyset$. Thus, $W_{n-1} \geq -a_{m-R_n}$ almost surely for all $n \geq 1$. The same arguments apply for black balls. $\square$

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MARKUS KUBA, INSTITUTE OF APPLIED MATHEMATICS AND NATURAL SCIENCES, UNIVERSITY OF APPLIED SCIENCES-TECHNIKUM WIEN, HÖCHSTÄDTPLATZ 5, 1200 WIEN

E-mail address: kuba@technikum-wien.at

HENNING SULZBACH, SCHOOL OF COMPUTER SCIENCE, MCGILL UNIVERSITY, 3480 UNIVERSITY STREET, H3A 0E9 MONTREAL, QC, CANADA

E-mail address: henning.sulzbach@gmail.com